Journal of Statistical Physics, Vol. 41, Nos.3/4, 1985

The Crossover from Classical to Quantum Regime in the Problem of the Decay of the Metastable State

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Received January 28, 1985

The probability of the decay of the metastable state has been found as a function of viscosity and temperature. At low temperatures, a classical overbarrier transition changes to the quantum tunneling. At low viscosity in classical and transitive semiquantum region of temperatures a depopulation of the distribution function is significant. The distribution function is shown to satisfy the integral kinetic equation, the kernel of which equals the transition probability. The probability of transitions, induced by the red noise, with the frequency comparable to the transition frequency, is found.

KEY WORDS: Quantum noises; decay probability; crossover region; depopulation of the distribution function.

1. INTRODUCTION

Past years have witnessed a renewed interest in the study of the quantum particle of motion, interacting with a heat bath. That was mainly stimulated by studies of superconducting tunnel junctions at low temperatures. For junctions of small size the lifetime of the metastable current state is not large and can be measured experimentally. At the given potential barrier the lifetime of metastable state depends on temperature and viscosity. At high temperatures the decay proceeds via a classical overbarrier transition. At low temperatures, the quantum tunneling is essential. At zero temperature the tunneling occurs from the ground state. As the temperature rises, the tunneling from higher and still higher energy levels becomes the most probable one. As one passes to the classical overbarrier transition, either first- or the second-order phase transition takes place. The

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0022-4715/85/1100-0425$04.50/0

phase transition order depends on the shape of the potential barrier. The phase transition is of the first order for potentials with a flat top (close to a rectangular barrier). In this case the quantum tunneling proceeds from deep levels at all temperatures, as far as it is significant. The total probability of a decay for such a potential is a sum of probabilities of the quantum tunneling and probability of the classical overbarrier transition. Depending on temperature, one or another mechanism is predominant. In many cases, in the vicinity of the lability point, the potential has a form of cubic parabola. For a cubic parabolic potential, as well as for one describing the tunnel contact, the second-order transition takes place. In this case, as temperature rises, the tunneling transition would take place from the higher and still higher energy level, which reaches the height of the potential barrier at the transition temperature T_0. For sufficiently high potential barrier, the decay rate Γ is exponentially small:

$$\Gamma = B \exp(-A) \tag{1}$$

The exponents A and the preexponential factor B depend on temperature and viscosity, and one can distinguish in the plane (T, η) several domains, corresponding to different regimes (Fig. 1):

At temperatures $T > T_0$, the Arrenius law holds:

$$A = U/T \tag{2}$$

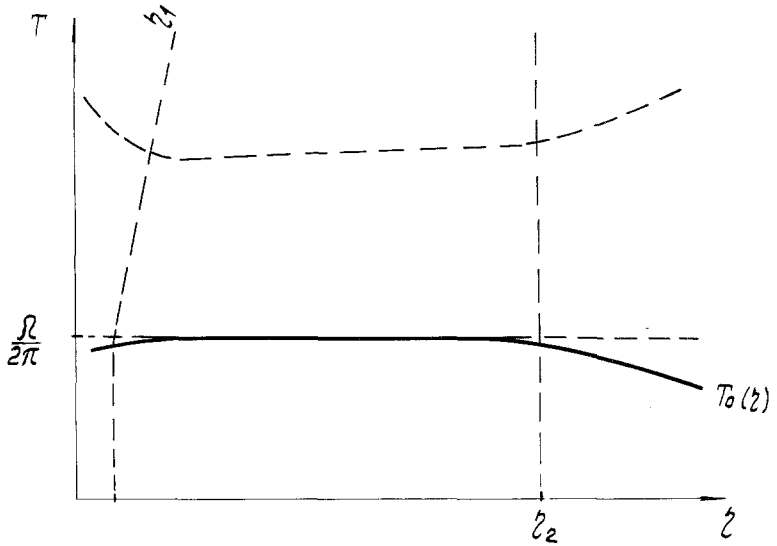

Fig. 1. The regions on (η, T) plane with different dependence of the exponent and the prefactor.

where U is the potential barrier height. At $T = T_0$ the function $A(T)$ has a discontinuous second derivative.[1] At $T < T_0$ the exponent $A(T)$ is a smooth function of temperature and tends to the constant limiting value at $T \to 0$. For example, in the limit of high viscosity and the cubic parabolic potential,[1]

$$A(T) = \frac{U}{T_0}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{T}{T_0}\right)^2\right] \tag{3}$$

In the narrow region near T_0 with width $T_0(T_0/U)^{1/2}$ fluctuations smear the singularity out. In this region

$$\Gamma(t) = 0.5\mathscr{B}[1 - \phi(x)] \exp(x^2 - U/T) \tag{4}$$

$$\phi(x) = 2/\pi^{1/2} \int_0^x dt \exp(-t^2)$$

where $x = \lambda(T - T_0)$. Coefficients λ, $\mathscr{B}$ and temperature T_0 depend on viscosity and were found in Refs. 2, 3, and 4. The decay probability has as a function of viscosity two characteristic points $\eta_{1,2}$:

$$\eta_1 = m\Omega T/U, \qquad \eta_2 = m\Omega \tag{5}$$

where Ω is the frequency of oscillations in the overturned potential

$$\Omega = (-U''/m)^{1/2} \tag{6}$$

m is particle mass and U'' the second derivative in the maximum position.

At viscosities $\eta \sim \eta_1$ the energy loss δ per period is of the temperature order T. At $\eta \gtrsim \eta_2$, the energy loss $\delta \sim U$ and classical motion becomes an aperiodical one. In the classical region of high temperatures $T \gg T_0$ preexponential factor found in Kramers' work[5] in the limiting cases of $\eta \gg \eta_1$ and $\eta \ll \eta_1$ and by Melnikov[6] at $\eta \sim \eta_1$.

The region of intermediate temperatures $T \gtrsim T_0$ where a transition from the classical to the quantum decay law[4] takes place, has been studied in Ref. 2 at $\eta_1 \ll \eta \ll \eta_2$ and in Ref. 7 at $\eta \gtrsim \eta_2$. In the present work we study the region of viscosities $\eta \gtrsim \eta_1$ and temperatures $T > T_0$. The limiting case of $\eta < \eta_1$ has been considered in Ref. 4. At temperatures $T < T_0$ the decay probability Γ may be written as follows:

$$\Gamma = 2 \operatorname{Im} \mathscr{F} \tag{7}$$

where $\mathscr{F}$ is the free energy.

At $T > T_0$ and $\eta \gg \eta_1$ the following formula is true:

$$\Gamma = 2(T_0/T)\,\mathrm{Im}\,\mathscr{F} \tag{8}$$

In the considered region these formulas are wrong because they both assume that the distribution function is an equilibrium one. However, at small viscosity $\eta \ll \eta_2$ probability Γ may be written in the following way:

$$\Gamma = \int \frac{dE}{2\pi} N(E)\,\gamma(E) \Big/ \sum_i N(E_i) \tag{9}$$

where $N(E)$ is the distribution function of quantum particle in energy and $\gamma(E)$ is the tunneling probability. Formula (9) gives the same result for the lifetime Γ as formulas (7) and (8) provided that the distribution function $N(E)$ in formula (9) takes its equilibrium value.[2] As will be seen below, such a change at $T > T_0$ is possible only in the region of viscosity values $\eta \gg \eta_1$. In region $\eta \gtrsim \eta_1$ it is necessary to take into account depopulation of the distribution function $N(E)$ at energies E close to U caused by the decay processes.

The distribution function $N(E)$ is defined by the transition processes between quasiclassical levels caused by the quantum noise. In the second part we shall get a common formula for such transitions probability. This formula is of particular interest and represents the main methodical achievement of the present paper. In the third part, the equation for the distribution function $N(E)$ has been solved and the expression for the decay probability Γ of metastable state has been derived. In the fourth part, the limiting case of small viscosity has been studied, and in the fifth the obtained results are used for the calculation of the lifetime of the current state of Josephson junction.

2. THE TRANSITION PROBABILITY CAUSED BY THE QUANTUM NOISES

It is comparatively simple to describe the motion of a quasiclassical particle in the field of noises in the two limiting cases. If the interaction of the particle with the field of noises is weak, then the transition probability per period is small and may be found with the help of nonstationary perturbation theory. If a noise can be regarded as white noise, then the particle behavior may be described with the help of the Fokker–Planck equation. Below, for the study of region $\eta \sim \eta_1$ and $T \gtrsim T_0$ we shall get the equation for the transition probability which in limiting cases transforms either into perturbation theory ($\eta \ll \eta_1$) or into the solution of the Fokker–Planck equation ($T \gg T_0$).

The transition temperature at $\eta \sim \eta_1$ is equal to

$$T_0 = \Omega/2\pi \tag{10}$$

where Ω is defined by formula (6).

If the width of the potential well is of the same order as the width of the potential barrier, then the characteristic time of the particle motion in the well, during which the particle passes almost the whole well, is of the order of Ω^{-1}. Usually, the noise frequency is of the order of temperature. At $T \sim T_0$ they are of the order of frequency of motion. In this case the noise cannot be considered as white noise and the Fokker–Planck equation cannot be used. The motion of the particle, interacting with a non-white noise has been studied in Refs. 8 and 9.

In all these works the differential equation, describing the diffusion in energetic variable, has been studied.

In the considered case, the characteristic transition energies are of order of temperature T_0. The distribution function at such changes of energy change rapidly and must be described by the integral equation.

If the energy of the particle is close to the height of the potential barrier, then the particle approaches the top of the barrier during a logarithmically long time. Near the top of the barrier it moves with a small velocity and small dissipation. This leads to independence of dissipative processes in different periods of motion. The processes of tunneling and dissipation do also decouple and it is possible to introduce the average transition probability per period of classical motion. In this chapter we derive this transition probability.

Suppose, that the Hamiltonian $\hat{H}$ of one-dimension quantum particle, interacting with a thermal bath, may be represented in the following way:

$$\hat{H} = \hat{H}(\varphi) + \varphi Q + \hat{H}_T(Q)$$

where $\hat{H}(\varphi)$ is the Hamiltonian of the particle with coordinate φ, moving in potential $U(\varphi)$, and $\hat{H}_T(Q)$ is the Hamiltonian of the thermal bath. The transition amplitude A_{if} from state i to state f of Hamiltonian $\hat{H}(\varphi)$ for time t is equal to

$$A_{if} = \left\langle f \left| \hat{T} \exp\left(-i \int_0^t \hat{\varphi}(t)\, \hat{Q}(t)\, dt \right| i \right) \right\rangle \tag{11}$$

where $\hat{T}$ is the time ordering operator and $\hat{\varphi}(t)$, $\hat{Q}(t)$ are operators in the interaction representation. For the matrix elements of operator $\hat{\varphi}$ we shall use quasiclassical formula

$$\langle f | \hat{\varphi}(t) | i \rangle = \frac{\omega}{2\pi} \oint d\tau\, \varphi_i(\tau) \exp[-i(E_f - E_i)(\tau - t)] \tag{12}$$

where $\varphi_i(\tau)$ is the value of coordinate φ at the moment of time τ on the classical trajectory with energy E_i, ω is the frequency of classical movement, and $\oint$ means integration over the period of classical motion. Quasiclassical matrix elements decrease quickly with the energy difference $|E_f - E_i|$ and are smooth functions of energy E_i. Let us take Fourier transforms of matrix element A_{if} over energy $E_f - E_i$:

$$A(s) = \sum_f A_{ij} \exp[i(E_f - E_i)s] \tag{13}$$

Expanding expression (11) in series of degrees of interaction, we shall get

$$\begin{aligned} A(s) = \sum_f \exp[i(E_f - E_i)s] \Big\{ & \delta_{if} \\ & - \frac{i\omega}{2\pi} \oint d\tau \int_0^t dt_1 \, \varphi_i(\tau) \, \hat{Q}(t_1) \exp[-i(E_f - E_i)(\tau - t_1)] \\ & - \left(\frac{\omega}{2\pi}\right)^2 \sum_n \oint d\tau_1 \oint d\tau_2 \, \varphi_n(\tau_1) \, \varphi_i(\tau_2) \int_0^t dt_1 \int_0^{t_1} dt_2 \, \hat{Q}(t_1) \, \hat{Q}(t_2) \\ & \times \exp[-i(E_f - E_n)(\tau_1 - t_1) - i(E_n - E_i)(\tau_2 - t_2)] + \cdots \Big\} \end{aligned} \tag{14}$$

By taking into account that energies E_n in formula (14) are close to E_i, it is possible to change all φ_n for φ_i. After that, the series in formula (14) are summed and we shall get the following expression for quantity $A(s)$:

$$A(s) = \hat{T} \exp\left[-i \int_0^t \varphi_i(s + t_1) \, \hat{Q}(t_1) \, dt_1 \right], \tag{15}$$

$$A_{if} = \frac{\omega}{2\pi} \oint ds \, A(s) \exp[-is(E_f - E_i)]$$

The probability of transition from state i into state f during the time t, averaged over the state of thermal bath, is equal to

$$W_{if} = \langle |A_{if}|^2 \rangle \tag{16}$$

We shall suppose noise to be Gaussian. In this case the average in formula (16) is expressed through a pair correlators of operator $\hat{Q}(t)$:

$$W_{if}=\left(\frac{\omega}{2\pi}\right)^2 \oint ds_1 \oint ds_2 \exp[-i(s_1-s_2)(E_f-E_i)]$$
$$\times \exp\left\{-\frac{1}{2}\iint_0^t [\varphi(s_1+t_1)\,\varphi(s_1+t_2)\,\mathscr{D}_{11}(t_1,t_2)\right.$$
$$+\varphi(s_2+t_1)\,\varphi(s_2+t_2)\,\mathscr{D}_{22}(t_1,t_2)-\varphi(s_1+t_1)\,\varphi(s_2+t_2)\,\mathscr{D}_{21}(t_1,t_2)$$
$$\left.-\varphi(s_2+t_1)\,\varphi(s_1+t_2)\,\mathscr{D}_{12}(t_1,t_2)]\,dt_1\,dt_2\right\} \tag{17}$$

where

$$\mathscr{D}_{11}(t_1,t_2)=\langle \hat{T}\hat{Q}(t_1)\,\hat{Q}(t_2)\rangle, \qquad \mathscr{D}_{22}(t_1,t_2)=\langle \hat{T}^{-1}\hat{Q}(t_2)\,\hat{Q}(t_2)\rangle$$
$$\mathscr{D}_{12}(t_1,t_2)=\langle \hat{Q}(t_1)\,\hat{Q}(t_2)\rangle, \qquad \mathscr{D}_{21}(t_1,t_2)=\langle \hat{Q}(t_2)\,\hat{Q}(t_1)\rangle \tag{18}$$

Correlation functions $\mathscr{D}$ depend only on the time difference and we shall suppose that they vanish rapidly out of a narrow region $|t_1-t_2|\sim T^{-1}$.

Later on we shall need the transition probability for a time period equal to the period of motion. Integrals in time in the exponent in formula (18) are carried on the period of classical motion. This circumstance leads to some simplifications. A rapid decrease of $\mathscr{D}$ functions enables us to spread the integration on the time difference t_1-t_2 in the exponent of formula (18) to infinite limits. In this approach the exponent of formula (18) is only the function of difference $s=s_2-s_1$. Take into account the property of correlation functions:

$$\mathscr{D}_{11}(t,t_1)+\mathscr{D}_{22}(t,t_1)-\mathscr{D}_{12}(t,t_1)-\mathscr{D}_{21}(t,t_1)=0 \tag{19}$$

Then transition probability per period of motion, given by formula (17), can be written in the form

$$W_{if}=\frac{\omega}{2\pi}\oint ds\; W(s)\exp[is(E_f-E_i)] \tag{20}$$

$$W(s)=\exp[\mathscr{W}(s)-\mathscr{W}(0)] \tag{21}$$

where

$$\mathscr{W}(s)=\sum_f \mathscr{W}_{if}\exp[-is(E_f-E_i)] \tag{22}$$

Quantity $\mathscr{W}_{if}$ is the transition probability per period from state i into state f, calculated in the perturbation

$$\mathscr{W}_{if}=\frac{2\pi}{\omega}|\langle f|\,\varphi\,|i\rangle|^2\,\mathscr{D}(E_f-E_i)$$

where matrix elements are defined by formula (12):

$$\mathscr{D}(E) = \tfrac{1}{2}(\mathscr{D}_{21}(E) + \mathscr{D}_{12}(-E))$$

If a thermal bath is in a thermal equilibrium and the interaction of the particle with it leads to the appearance of viscosity with viscosity coefficient η, then we have[10]

$$\mathscr{D}(E) = \eta \left[E \coth\left(\frac{E}{2T}\right) - E \right] \tag{23}$$

In the potential $U(\varphi)$ having the form of a cubic parabola,

$$U(\varphi) = 3U(\varphi/\varphi_0)^2 \left(1 - \frac{2}{3}\frac{\varphi}{\varphi_0}\right) \tag{24}$$

function $\mathscr{W}(s)$ is equal to

$$\mathscr{W}(s) = \frac{18\pi\varphi_0^2}{\Omega^4} \int_{-\infty}^{\infty} dE \frac{E^2 \mathscr{D}(E)}{\sin h^2(\pi E/\Omega)} \exp(-isE)$$

Note that if the thermal bath is in a thermal equilibrium state, then the following condition is fulfilled:

$$\mathscr{D}(E) = \mathscr{D}(-E) \exp(-E/T)$$

So we get

$$\mathscr{W}(i/T) = \mathscr{W}(0), \qquad W(i/T) = W(0) = 1$$

In the limiting case of small viscosity ($\eta \ll \eta_1$) the exponent in formula (21) is small and the transition probability may be found according to the pertubution theory. At high temperature $T \gg \Omega$ in formula (20) small values $s \sim T^{-1}$ are essential. So expression (22) can be written in the following form:

$$\mathscr{W}(s) = \mathscr{W}(0) + \delta(is - s^2 T) \tag{25}$$

where δ is the energy loss by the particle per period,

$$\delta = -\sum_f \mathscr{W}_{if}(E_f - E_i) = \eta \oint dt(\partial\varphi/\partial t)^2 \tag{26}$$

Substituting expression (25) into formula (20), we shall get

$$W_{if} = \frac{\omega}{2\pi}\left(\frac{\pi}{\delta T}\right)^{1/2} \exp\left[-\frac{(\delta + E_f - E_i)^2}{4\delta T} \right]$$

The same expression for W_{if} may be obtained from the Fokker–Planck equation.

At temperature $T \sim \Omega$ essential values of s in formula (20) are not small and the transition probabilities W_{if} do not follow the Gaussian distribution.

3. THE DECAY PROBABILITY OF THE METASTABLE STATE

In the considered case of small viscosity $\eta \ll \eta_2$ the decay probability of the metastable state is given by formula (9), in which the penetration factor may be found from the quantum mechanical formula, neglecting the dissipation process. For energies, close to the potential barrier height U, the barrier is parabolic and the penetration factor γ is equal to

$$\gamma(E) = [1 + \exp(-2\pi E/\Omega)]^{-1} \tag{27}$$

Hereafter, energy E is counted from the top of the potential barrier. In formula (9), $N(E)$ is the distribution function of the particles that are near the barrier and move toward the barrier. It is connected with the distribution function of particles $N^{(-)}(E)$ reflected from the barrier, by the following equations:

$$N^{(-)}(E) = [1 - \gamma(E)]\, N(E) \tag{28}$$

$$N(E_f) = \sum_i W_{if}\, N^{(-)}(E_i)$$

As was noted above, for energies E close to the top of the barrier, the period of a classical motion is logarithmically large, which means that the distance between levels is small and the sum over states in formula (28) may be substituted for the integral.

Using the formulas (20) and (27), the equation for the distribution function $N^{(-)}(E)$ takes the form

$$[1 + \exp(2\pi E/\Omega)]\, N^{(-)}(E_f) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \int_{-\infty}^{\infty} ds\ W(s)\, N^{(-)}(E) \exp[is(E_f - E)] \tag{29}$$

The solution of the equation (29) can be found with the method, given in Ref. 6. In this work the Gaussian distribution of probability W, corresponding to a white noise, i.e., to high temperatures, has been considered.

Let us consider the Laplace transform of the distribution function $N^{(-)}(E)$,

$$n(p)=\int_{-\infty}^{\infty} dE\, N^{(-)}(E)\exp(-pE),$$
$$N^{(-)}(E)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dp\, n(p)\exp(pE) \tag{30}$$

At large negative values of the energy function $N^{(-)}(E)$ tends to its equilibrium value $\sim\exp(-E/T)$. This leads to the condition for parameter α in formula (30) $\alpha<-1/T$. From the equation (29) for function $n(p)$ we obtain the equation

$$n(p-2\pi/\Omega)=\hat{K}(p)\,n(p)=-[1-W(-ip)]\,n(p) \tag{31}$$

The equation of the type (31) with function $\hat{K}(p)$, analytical in the left of right half-plane and tending to 1 at infinity, has the solution that can be written as an infinite product. In a general case, it is necessary to represent function $\hat{K}(p)$ as a product of two functions: one analytical in the left plabe and the other in the right one:

$$\hat{K}(p)=[1-W(-i\infty)]\frac{p-\beta}{-p+\varkappa}\cdot\frac{K^{+}(p)}{K^{-}(p)} \tag{32}$$

where

$$K^{\pm}(p)=\exp\left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dp_1}{p_1-p\pm\nu}\ln\left[\frac{1-W(-ip_1)}{1-W(-i\infty)}\cdot\frac{p_1-\varkappa}{p_1-\beta}\right]\right\} \tag{33}$$

Kernel $K^{+}(p)$ is analytical in the half-plane $\mathrm{Re}\,p<\alpha$ whereas $K^{-}(\rho)$ is analytical in the half-plane $\mathrm{Re}\,p>\alpha$, parameters $\varkappa,\beta$ satisfy conditions $\varkappa<\alpha$, $\beta>\alpha$. The factor $(p-\varkappa)/(p-\beta)$ is introduced to compensate for the phase factor, acquired by function $\hat{K}(p)$ along the integration contour. The solution of the equation (31) we search for in the form of the product of the solutions equations with kernels, equal to separate cofactors in formula (32). As a result we obtain

$$n(p)=\frac{CK^{-}(p)\,\Gamma(1+(p-\varkappa)\Omega/2\pi)\,\Gamma(-(\rho-\beta)\Omega/2\pi)}{\exp[((\beta-\varkappa)\Omega/2\pi)\ln N]\prod_{n=0}^{N}K^{-}(p+2\pi n/\Omega)\,K^{+}(p-2\pi n/\Omega)}$$
$$\times\exp\left\{-\frac{p\Omega}{2\pi}\ln[1-W(-i\infty)]\right\},\qquad N\to+\infty \tag{34}$$

Expression (34) for $n(p)$ does not depend on the choice of parameters $\varkappa,\beta$. We put $\beta=-1/T$. At such a value of parameter β the zero of the ker-

nel $1-W(-ip)$ at the point $p=-1/T$ is eliminated, and the integration contour may be shifted into the position $(-1/2T-i\infty;\ -1/2T+i\infty)$. After that, putting $\varkappa=-1/T$, the expression for $n(p)$ takes the form

$$n(p)=\frac{C}{\sin[(p+1/T)\Omega/2]} \times \exp\left\{\frac{i\Omega}{4\pi}\int_{-1/2T-i\infty}^{-1/2T+i\infty} dp_1 \ln[1-W(-ip_1)] \times \left[\cot\left(\frac{(p_1-p)\Omega}{2}\right)-\cot\left(\frac{(p_1+1/T)\Omega}{2}\right)\right]\right\} \tag{35}$$

At large negative energies E the behavior of $N(E)$ is governed by the pole of the function $n(p)$ at $p=-1/T$

$$N(E)=-\frac{2C}{\Omega}\exp\left(-\frac{E}{T}\right) \tag{36}$$

It is convenient to take constant C such that $N(E)$ would be normalized to a unity. Below, we shall suppose viscosity η to be not too small, so that the thermal equilibration rate m/η exceds the metastable state decay rate. If so, then the distribution function is given by formula (36) near the bottom of the potential well. Considering, that the potential near the bottom of the well is a parabolic one,

$$U(\varphi)=-U+m\Omega_0^2(\varphi-\varphi_0)^2/2 \tag{37}$$

one finds for coefficient C the following expression:

$$C=-\Omega\sinh(\Omega_0/2T)\exp(-U/T) \tag{38}$$

Formulas (28), (30), (35), and (38) allow one to find the distribution function $N(E)$. Substituting it into formula (9) we shall find the probability of the decay of the metastable state Γ. For the probability of tunneling, defined by formula (27), Γ is directly expressed through $n(p)$:

$$\Gamma=\frac{n(-2\pi/\Omega)}{2\pi}, \qquad \Gamma=\frac{\Omega}{2\pi}\frac{\sinh(\Omega_0/2T)}{\sin(\Omega/2T)}\exp\left(-\frac{U}{T}\right)Y \tag{39}$$

$$Y=\exp\left\{\frac{i\Omega}{4\pi}\int_{-1/2T-i\infty}^{1/2T+i\infty} dp\ln[1-W(-ip)] \times\left[\cot\left(\frac{p\Omega}{2}\right)-\cot\left(\frac{(p+1/T)\Omega}{2}\right)\right]\right\}$$

At high temperature $T \gg \Omega$ function $W(s)$ is given by formulas (21), (25) and Eq. (39) coincides with the result of Refs. 6 and 5. At $T - T_0 \ll T_0$, in the leading approximation, $Y = 1$ and the expression (39) and asymptotics of formula (4) at $T - T_0 \gg T_0(T_0/U)^{1/2}$ do coincide. In order to obtain formula (4), it is necessary to take into account the deviation of the potential from the parabolic one. At viscosity $\eta \gg \eta_1$ the function $W(s)$ on the integration contour is exponentially small, because $\mathscr{W}(s)$ is large. In the zeroth approximation $Y = 1$ and formula (39) will coincide with that of Ref. 2. In this case the corrections are exponentially small and can be found by the saddle point method

$$Y = 1 - \frac{\Omega}{2\pi}\left(-\frac{2\pi}{(\partial^2 \mathscr{W}/\partial s^2)_{i/2T}}\right)^{1/2} \cot\left(\frac{\Omega}{4T}\right) \exp[-\mathscr{W}(0) + \mathscr{W}(i/2T)] \tag{40}$$

While deriving formula (40) we have explicitly made use of the extremum of function $\mathscr{W}(s)$ for the equilibrium thermal bath being at $s = i/2T$.

4. THE CASE OF SMALL VISCOSITY ($\eta \ll \eta_1$)

At a small viscosity $\delta T/T_0^2 \ll 1$ the quantity $W(s)$ is close to a unity, and the difference $1 - W(s)$ is proportional to viscosity η. Therefore, the dependence on viscosity η in formula (39) for Y can be explicitly separated:

$$Y = (\delta/T)^{1 - T_0/T} f(T_0/T) \tag{41}$$

where δ is the energy loss per period, given by formula (26), and function $f(T_0/T)$ is equal to

$$f\left(\frac{T_0}{T}\right) = \exp\left\{\frac{i\Omega}{4\pi}\int_{-1/2T - i\infty}^{-1/2T + i\infty} dp \ln\left\{\frac{T}{\delta}[\mathscr{W}(0) - \mathscr{W}(-ip)]\right\}\right.$$
$$\left. \times \left[\cot\left(\frac{p\Omega}{2}\right) - \cot\left(\frac{(p + 1/T)\Omega}{2}\right)\right]\right\} \tag{42}$$

At $T \gg T_0$ in formula (42) small values of p are essential and for $\mathscr{W}(-ip)$ formula (25) can be used. As a result $f(0) = 1$.

Thus, in the classical limit of high temperatures the expression for Γ [formulas (39) and (42)] coincides with Kramer's result. It should be noted, that a classical limit holds only in the region $T \gg T_0 \ln(T/\delta)$. At $T \to T_0$ function $f(1) = 1$. The values of function $f(x)$ for the cubic poten-

tial (24) in the intermediate region are given in Table I. For a small value of parameter T_0/T function Y is equal to

$$Y=\left(\frac{\delta}{T}\right)^{1-T_0/T}\exp\left\{-\frac{2T_0}{T}\ln\left(\frac{2T}{3T_0}\right)-\frac{5}{8\pi^2}\frac{\delta}{T_0}\right\}, \qquad \frac{\delta T}{T_0^2}\ll 1$$

$$T=\left(\frac{\delta}{T}\right)\exp\left\{\left(\frac{\delta}{\pi T}\right)^{1/2}\xi(1/2)\right\}, \qquad \frac{\delta T}{T_0^2}\gg 1$$

where $\xi(1/2)=-1.46035$ is the Riemann zeta function.

In contrast to a classical Kramer formula a small parameter in formula (41) enters with exponents, smaller than 1. This is due to the depopulation of the distribution function at energies close to the potential barrier height, caused by a small dissipation. Then main contribution comes from the states with energy E_c, at which the tunneling probability is equal to the probability of transition, induced by friction[11]

$$\gamma(E_c)=\mathscr{W}(0)\sim\delta T/\Omega^2 \tag{43}$$

The depopulation of the distribution function $N(E)$ at $E>E_c$ is evident from simple physical considerations and explicitly follows from formulas (30) and (35).

We have supposed in the above that the energy levels lying near the top of the potential barrier, where the potential has a parabolic form, are

Table I. The Functions $f(T_0/T)$ and $Y(\delta/T, T_0/T)$

δ/T		0.25	0.5	1	2	4	6
T_0/T	$f(T_0/T)$	Y	Y	Y	Y	Y	Y
0	1	0.1658	0.2802	0.443	0.6409	0.8299	0.9126
0.05	0.7684	0.1666	0.2808	0.4434	0.641	0.8298	0.9125
0.1	0.6734	0.1689	0.2828	0.4447	0.6414	0.8296	0.9122
0.15	0.6179	0.173	0.2863	0.447	0.6422	0.8293	0.9117
0.2	0.5835	0.1793	0.2917	0.4506	0.6437	0.8291	0.9111
0.3	0.5511	0.1996	0.3102	0.4636	0.6496	0.8296	0.9101
0.4	0.5494	0.2318	0.3407	0.4869	0.6619	0.8325	0.9101
0.5	0.5698	0.2787	0.3857	0.5231	0.6835	0.8399	0.9123
0.6	0.6097	0.3449	0.4482	0.5743	0.7164	0.8536	0.9181
0.7	0.6694	0.4372	0.5329	0.6432	0.7624	0.8753	0.9287
0.8	0.7514	0.566	0.6462	0.7335	0.8234	0.9062	0.9453
0.9	0.8596	0.7463	0.7976	0.8502	0.9017	0.9474	0.9688
1	1	1	1	1	1	1	1

the most significant ones. At a small viscosity the significant energy values E are rather close to E_c, and the potential cannot be considered parabolic. For such energies

$$\gamma(E)=\gamma(E_c)\exp\left[\frac{2\pi}{\Omega_c}(E-E_c)\right] \tag{44}$$

where Ω_c is the frequency of the classical motion at $E=E_c$ in the overturned potential. For such a small viscosity it is necessary to make substitution $\Omega\to\Omega_c$ in formulas (39) and (41).[4] The temperature transition from classical to quantum decay regime decreases with the decreasing viscosity and is given by

$$T_0=\frac{\Omega_c}{2\pi}=\frac{\Omega}{2\pi}\left[1-\frac{1}{2\pi}\frac{\partial\Omega}{\partial E}\ln\left(\frac{1}{\mathscr{W}(0)}\right)\right] \tag{45}$$

The last equality in formula (45) holds only at a small deviation of frequency Ω_c from Ω. At temperature T close to T_0 the dependence on energy of the two factors in formula (9) is mutually compensate. For this reason a wide region of energies is essential and in formula (44) it is necessary to keep in the exponent the next-to-leading term of the expansion in $E-E_c$. This would result in formula (4).

5. THE DECAY OF THE CURRENT STATE IN TUNNEL JUNCTION

The behavior of the superconducting tunnel junction with large capacitance is similar to that of the quantum particle with mass $m=C/e^2$ (C is the junction capacitance) moving in the potential

$$U(\varphi)=-\frac{\mathscr{J}}{e}\varphi-\frac{\mathscr{J}_c}{2e}\cos(2\varphi) \tag{46}$$

where $\mathscr{J}$ is the current into the junction and 2φ the phase difference between two superconductors; $\mathscr{J}_c$ is the critical current of the contact. It follows from formula (45) that the oscilation frequencies Ω, Ω_0 near overturned potential and near the potential bottom are equal and are derived by thez following equation:

$$\Omega^2=\Omega_0^2=\frac{2e\mathscr{J}_c}{C}\left[1-\left(\frac{\mathscr{J}}{\mathscr{J}_c}\right)^2\right]^{1/2} \tag{47}$$

The dissipative processes enter through the interaction of the collective variable φ with normal excitations. The transition probability $\mathscr{W}_{if}$ between the levels of potential (45) is equal to[4]

$$\mathscr{W}_{if} = \frac{\pi}{R_N e^2 \omega} \int d\varepsilon \left[1 - \tanh\left(\frac{\varepsilon}{2T}\right)\right]\left[1 + \tanh\left(\frac{\varepsilon - E}{2T}\right)\right]$$
$$\times \left[\rho_L(\varepsilon)\,\rho_R(\varepsilon - E)\,|\langle f|\exp(i\varphi)|i\rangle|^2 - \mathscr{F}_L^-(\varepsilon)\,\mathscr{F}_R^-(\varepsilon - E)\right.$$
$$\left.\times\,(\langle f|\exp(i\varphi)|i\rangle^2 + \langle f|\exp(-i\varphi)|i\rangle^2)/2\right] \tag{48}$$

where $E = E_f - E_i$, indices L, R mean the left, accordingly the right superconductors. Functions ρ and $\mathscr{F}^-$ for the superconductors without paramagnetic impurities are equal to

$$\rho(\varepsilon) = \frac{|\varepsilon|\,\theta(|\varepsilon| - \Delta)}{(\varepsilon^2 - \Delta^2)^{1/2}}, \qquad \mathscr{F}^-(\varepsilon) = \Delta\rho(\varepsilon)/\varepsilon \tag{49}$$

If the contact is not shunted by the normal resistance then the dissipative processes are exponentially small $\sim\exp(-\Delta/T)$ and the case of a small viscosity can be easily realized. The formulas, derived above, are also applicable to the tunnel junction, if in formula (22) for $\mathscr{W}_{if}$ the expression (48) is used. In particular, the transition temperature T_0 between the quantum and classical regimes is defined by formula (45) and frequency Ω is given by formula (47). At $T \gg T_0$ the energy loss δ per period for the identical superconductors are equal to

$$\delta = \frac{1}{Re^2} \int d\tau \left(\frac{\partial\varphi}{\partial\tau}\right)^2 \left[1 + \left(1 - \frac{R}{R_{\text{sh}}}\right)\cos(2\varphi)\right] \tag{50}$$

where contact resistance R is equal to

$$R^{-1} = R_{\text{sh}}^{-1} + \frac{2e\mathscr{I}_c}{\pi T}\exp\left(-\frac{\Delta}{T}\right)\ln\left(\frac{T}{T_0}\right) \tag{51}$$

In formula (51) R_{sh} is the shunt resistance. At $T \sim T_0$ the frequency dependence of the contact resistance to the normal current becomes essential. If the contact is shunted by the normal metal, then that dispersion is small and the correlation function of noises has a form of (23). For current $\mathscr{I}$, close to a critical one, the potential energy has a form of cubic parabola. The dependence of lifetime of metastable state Γ^{-1} on temperature and viscosity is defined by formulas (41) and (42) and is given in Table I. The

transition to the case of small viscosity $\eta < \eta_1$ is realized at the contact resistance, exceeding the quantum limit

$$Re^2 > 1 - (\mathscr{I}/\mathscr{I}_c)^2 \tag{52}$$

APPENDIX A

The results, obtained in this paper, allow us to find the lifetime of metastable state in the intermediate region of viscosity values $\eta \gtrsim \eta_1$ and temperatures $T \sim T_0$. In this region, as at $T < T_0$, the preexponential factor in the expression for Γ depends on the specific form of the potential. The numeral results are given for the potential, having the form of cubic parabola. Such a potential does definitely occur near the lability point of the disappearance of the metastable state. For the potentials of different form the problem is reduced to the integrals from function $\varphi(\tau)$ on a classical trajectory with the "particle" energy, equal to the height of the potential barrier.

Formula (39), complemented by the results of Refs. 1, 2, 3, and 6, gives a full description of decay probability Γ in the whole region of the variations of viscosity η and temperature $T \gtrsim T_0$:

$$\Gamma = 2\pi^2 m T^2 \left(\frac{\pi}{\tilde{B}}\right)^{1/2} Y T_0 \left(\frac{\Omega_0}{\Omega}\right) \frac{\Gamma(2-\chi_1)\,\Gamma(2-\chi_2)}{\Gamma(1-n_1)\,\Gamma(1-n_2)}$$

$$\times\ [1-\phi(x)] \exp\left(x^2 - \frac{U}{T}\right)$$

$$\chi_{1,2} = \frac{1}{2\pi T}\left[-\frac{\eta}{2m} \pm \left(\frac{\eta^2}{4m^2} + \Omega^2\right)^{1/2}\right]$$

$$n_{1,2} = \frac{1}{2\pi T}\left[-\frac{\eta}{2m} \pm \left(\frac{\eta^2}{4m^2} - \Omega_0^2\right)^{1/2}\right]$$

$$x = 2\pi^2 m T^2 (1-\chi_1)(1-\chi_2)/\tilde{B}^{1/2}$$

$$\tilde{B} = T\left[\frac{U^{\mathrm{IV}}(\varphi_0)}{4} + \frac{[U^{\mathrm{III}}(\phi_0)]^2}{2m\Omega^2}\left(1 - \frac{\Omega^2}{2[(4\pi T)^2 - \Omega^2 + 4\pi T\eta/m]}\right)\right]$$

$$\chi_1 = T_0/T$$

Function $Y(\eta, T)$, found above, takes into account the depopulation of the distribution function. In the classical region $T \gg T_0$ it coincides with the result of Refs. 5 and 6 and differs from the interpolation results.[12,13]

Thus the previously unstudied regions in the plane (η, T) are exhausted (see Fig. 1).

It has been supposed in this paper that the correlation function has dispersion on frequancy $\tilde{\omega} \sim T$, and the potential, characterized by the two parameters: the height of the potential barrier U and the oscillation frequency, i.e., the barrier width and the potential well are of the same order. Both suppositions are fulfilled for tunnel contacts. In the study of other systems one should take into account that the received results are valid in a wider region, when the characteristic noises frequency satisfy to $\tilde{\omega} \gg \omega\gamma(E_c)$. Here ω is a logarithmically small frequency of a classical motion. Transparency $\gamma(E_c)$ is small in case of a small viscosity [formula (43)].

If the condition on frequency $\tilde{\omega}$, given above, is not fulfilled, then the transition to adiabatic limit occurs. The boundaries of the regions, described in Fig. 1, may be shifted, if the potential well radius and the barrier width do differ markedly.

APPENDIX B

Let us consider once again those assumptions which are the basis of this paper and let us note the connection of the results obtained in it with the results of other authors. The dissipation influence on the probability of quantum tunneling was first considered in the paper by Calderia and Leggett.[14] Case $T=0$ was considered in these papers. Low-temperature corrections to the decay probability were found in Ref. 15. In our paper we were interested in the temperature region $T \sim T_0$, where the crossover from quantum to classical decay regime occurs. At sufficiently large viscosity $\eta \lesssim \eta_2$ the decay probability was found in Refs. 3, 4, and 6. Some of these results were obtained later in Ref. 16. In the introduction these results are given to exhaust the description. In the present paper we studied the region of a small viscosity $\eta \sim \eta_1$. In this region viscosity affects weakly the quantum tunneling. However, in the temperature region $T \lesssim T_0$ viscosity has an effect upon the decay probability, because it defines the distribution function. The distribution function differs from the equilibrium one and formulas (7) and (8) are not available. In this region the viscosity processes are characterized by parameter δ, equal to the energy losses of a particle per period, while moving with energy, close to the height of the potential barrier (26). While moving in potential $U(\varphi)$, having the form of a cubic parabola (24), parameter δ is equal to

$$\delta = \frac{6 \cdot 6^{1/2}}{5} \eta \varphi_0 \left(\frac{U}{m}\right)^{1/2} \tag{53}$$

Evaluating parameter η_1 we supposed that the width of the potential well is of the same order as the barrier width. If the width of the classical

available region is large, then the value of parameter η_1 is diminished. In this case region $\eta_1 \ll \eta \ll \eta_2$ widens, where the distribution function is equilibrium and the results of Ref. 2 are available. The study of region $\eta \sim \eta_1$ is of great interest to us, because even in the classical region of high temperatures $T \gg T_0$ the results of different authors do not coincide. It was supposed in Ref. 12 that particles leave the well uniformly in time. It seems to us that this assumption is not valid and the results of Ref. 12 may be considered only as interpolating ones. We used the method of Ref. 6, evidently taking into account the circumstance that a particle leaves the well only approaching the barrier. The account of all that results in the system of Eqs. (28). At $T \gg T_0$ our results in the notations of Ref. 12 have the form

$$\frac{r}{r_{ts}} = \frac{2\pi T_0}{\Omega_p} \cdot Y = \left\{ 1 - \frac{G}{2[1-(F/V_0)^2]^{1/4}} \right\} Y\left(\frac{\delta}{T}\right)$$

$$\frac{\delta}{T} = \frac{36E_b}{5T} \frac{G}{[1-(F/V_0)^2]^{1/4}}$$

where values $Y(\delta/T)$ are given in the upper line of Table I. This expression coincides very well with the results of the numerical calculations, given in Ref. 12 for $F/V_0 = 0.985$, $E_b/T = 3.938...$.

At small values of parameter δ/T the first two terms of expansion of function $Y(\delta/T)$ coincide with the interpolating formula of Ref. 12 at $\alpha = -\zeta(1/2)/\pi^{1/2} = 0.824$. At large values of parameter δ/T asymptotic expansion (40) differs markedly from the results of Ref. 12. The numerical calculations are given in the assumption that the viscosity coefficient does not exhibit frequency dispersion. This assumption holds true for the tunnel junctions, shunted by normal resistance. We have supposed above that noises are distributed according to the Gaussian law. This assumption is fulfilled for the tunnel junctions, for the heat bath of harmonic oscillators and in the cases when the interaction with a heat bath is reduced to frequent but weak shocks.

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